ON THE FERMAT AND HESSIAN POINTS FOR THE NON-EUCLIDEAN TRIANGLE AND THEIR ANALOGUES FOR THE TETRAHEDRON.

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The euclidean triangle has two Fermat or equiangular points. A Fermat point F is defined projectively by the condition that the lines joining F to the absolute point pair I, J are the hessian of the triad of lines joining F to the vertices. If we take actual perpendiculars on the sides as coördinates x_1, x_2, x_3, the points F are transformed by the Desargues transformation $x = 1/y$ into a pair of points H, which are defined most simply by the condition that the feet of the perpendiculars from H on the sides form an equilateral triangle. The Fermat points may be defined in another way which will be found very suggestive. If we consider a particle acted on by three equal forces directed along lines through the vertices, the points F will be positions of equilibrium. We may substitute geometrical for dynamical ideas by considering the potential energy. If r_1, r_2, r_3 are the distances to the vertices the points F give the stationary values of $r_1 \pm r_2 \pm r_3$. The choices of sign, four in number, correspond to the different combinations of pushes and pulls, only two of the four having solutions.

In the non-euclidean plane our point pair I, J is replaced by the conic (line coördinates)

$$\Omega \equiv \xi_1^2 + \xi_2^2 + \xi_3^2 - 2c_1\xi_2\xi_3 - 2c_2\xi_1\xi_3 - 2c_3\xi_1\xi_2 = 0,$$

where the c's are the cosines of the internal angles, which in this case are independent. We will denote by Δ_Ω the discriminant of Ω.

Lines from a point x_1, x_2, x_3 to the vertices are

$$0,\ x_3,\ -x_2; \qquad -x_3,\ 0,\ x_1; \qquad x_2,\ -x_1,\ 0$$

and the hessian pair of this triad is $(1/x_1,\ \omega/x_2,\ \omega^2/x_3)$ and $(1/x_1,\ \omega^2/x_2,\ \omega/x_3)$ where ω is a complex cube root of unity. If x is a Fermat point, this pair touches Ω. The two equations thus obtained reduce to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{2c_3}{x_1x_2} = \frac{1}{x_1^2} + \frac{1}{x_3^2} + \frac{2c_2}{x_1x_3} = \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{2c_1}{x_2x_3}\,. \tag{1}$$

Writing $y = 1/x$ we get the pencil of conics

$$y_1^2 + y_2^2 + 2c_3y_1y_2 = y_1^2 + y_3^2 + 2c_2y_1y_3 = y_2^2 + y_3^2 + 2c_1y_2y_3. \tag{2}$$

The four base points of this pencil satisfy the definition of the hessian

points H. We thus see that in the non-euclidean plane there are four Fermat points and four hessian points. Noting that the equation $x_2^2 + x_3^2 + 2c_1x_2x_3 = 0$ gives the tangents from 1, 0, 0 to Ω and that each of the three conics is on a vertex, we see that each conic is on the vertices of a quadrilateral formed by two pairs of tangents and on the remaining vertex. If $\Delta_\Omega = 0$ two of the points H coincide with I and J and the conics are all circles. The Desargues transformation is in this case isogonal, I and J being interchanged. Thus in the euclidean plane the absolute point pair should in a sense be counted with both sets of points.

We pass now to the tetrahedron, and take as before actual perpendiculars on the faces as coördinates. The equation of the absolute is then

$$\Omega \equiv \Sigma\xi_i^2 + \Sigma 2c_{ij}\xi_i\xi_j = 0, \quad (i, j = 1, 2, 3, 4, \text{ and } i \neq j),$$

where c_{12} is the cosine of the interior angle between the planes 1 and 2, and the c's are all independent for the most general case in non-euclidean space. The definition for the analogue of the Fermat points is suggested by the dynamical conception outlined above. If four equal forces from x to the four vertices are in equilibrium, the bisector of the angle between any pair of forces bisects the angle between the remaining pair, and the two pairs are equally inclined to the bisector. This bisector meets a pair of opposite edges, and the three bisectors are mutually perpendicular. The configuration of four directions may be realized on a sphere by taking a point P and its three reflections in the vertices of a tri-rectangular spherical triangle. If we refer the tetrahedron to these three lines as rectangular cartesian axes it takes the form

$$\pi a, \pi b, \pi c; \qquad \rho a, -\rho b, -\rho c; \qquad -\sigma a, \sigma b, -\sigma c; \qquad -\tau y, -\tau b, \tau c.$$

We are thus led to seek, as the analogue of the Fermat points, those points S such that the lines from S to the pairs of edges are mutually perpendicular. In euclidean space, for which our dynamical conceptions hold, these points are also defined by the condition that $r_1 \pm r_2 \pm r_3 \pm r_4$ shall have stationary values. The more restricted problem of making $r_1 + r_2 + r_3 + r_4$ a minimum has been considered by Sturm,* who notes the orthogonality of the lines from such a point to the edges, but who obtains an analytic solution only as the intersection of three surfaces of the 12th order. Two problems in Wolstenholme's collection may also be noted. The first of these (No. 2024) deals essentially with the dynamical aspect, and the second (No. 2030) with finding points such that the lines to the edges are orthogonal. There is no indication of any connection between the two problems.

* *Crelle J.*, 97, p. 49 (1884).

The three planes

$$1/x_1,\ 1/x_2,\ -1/x_3,\ -1/x_4; \qquad 1/x_1,\ -1/x_2,\ 1/x_3,\ -1/x_4;$$
$$1/x_1,\ -1/x_2,\ -1/x_3,\ 1/x_4,$$

each contain two lines on the point x meeting pairs of edges. These planes are mutually perpendicular if

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{2c_{12}}{x_1x_2}=\frac{1}{x_3^2}+\frac{1}{x_4^2}+\frac{2c_{34}}{x_3x_4}, \qquad \text{etc.,} \tag{3}$$

equations which are given by Wolstenholme. If now we write $y = 1/x$ we get the net of quadrics

$$y_1^2 + y_2^2 + 2c_{12}y_1y_2 = y_3^2 + y_4^2 + 2c_{34}y_3y_4, \qquad \text{etc.,} \tag{4}$$

defining a set of eight points M* which have the property that the feet of the perpendiculars on the faces form an equifacial tetrahedron. They are thus similar to the hessian points of the triangle, so that the two sets S and M form a quite perfect analogue of the two-dimensional case.

The equations of the net (4) may be rewritten

$$\pi_1y_1 = \pi_2y_2 = \pi_3y_3 = \pi_4y_4,$$

$$\text{where } \pi_i = c_{i1}y_1 + c_{i2}y_2 + c_{i3}y_3 + c_{i4}y_4,\ c_{ii} = 1.$$

The points S thus go into the points M by the collineation

$$\pi_1/x_1 = \pi_2/x_2 = \pi_3/x_3 = \pi_4/x_4. \tag{5}$$

The two sets are thus projectively equivalent. The fixed points of the collineation are apolar to (Ω) and to the quadric

$$c_{12}\xi_1\xi_2 + c_{13}\xi_1\xi_3 + \cdots = 0 \tag{6}$$

which touches the faces at the feet of the altitudes. So far no condition has been placed on the c's. If the space is euclidean $\Delta_\Omega = 0$. This condition is equivalent to the coexistence of the equations

$$\begin{aligned} -A_1 + c_{12}A_2 + c_{13}A_3 + c_{14}A_4 &= 0,\\ c_{12}A_1 - A_2 + c_{23}A_3 + c_{24}A_4 &= 0,\\ c_{13}A_1 + c_{23}A_2 - A_3 + c_{34}A_4 &= 0,\\ c_{14}A_1 + c_{24}A_2 + c_{34}A_3 - A_4 &= 0, \end{aligned} \tag{7}$$

where the A's are the areas of the faces. Comparing these with (5) we see that the point and plane A_1, A_2, A_3, A_4 are fixed. The plane is however the plane at infinity, and the point is the "symmedian" point. The

* The discovery of these is due to Dr. F. D. Murnaghan, to whom I owe in its essentials the above elegant treatment, which replaces my own cumbrous solution.

collineation is thus affine, being a pure strain whose axes are the axes of the quadric (6).

The configuration of the eight points has not been in general determined. The equations of the net appear in a canonical form involving 6 constants, which is also the number of absolute invariants of a tetrahedron and quadric. The plane quartic obtained by equating to zero the discriminant of the net also appears in a canonical form, the terms in x^3y, etc., being absent. This would seem to indicate that the net is unrestricted, except when $\Delta_\Omega = 0$. The behavior of the net in special cases throws some doubt on this point, but the problem involves the little known subject of combinants of a net of quadrics, and must be postponed for the present.

Special cases will be considered only briefly. The most important of these is the equifacial tetrahedron (not necessarily euclidean), defined by $c_{14} = c_{23} \equiv c_1$, etc. Writing

$$\varphi_1 = -x_1^2 + x_2^2 + x_3^2 - x_4^2 - 2c_{23}x_2x_3 + 2c_{14}x_1x_4, \text{ etc.};\ s_1 = \sqrt{1 - c_1^2}, \text{ etc.},$$

the quadrics

$$\varphi_1/s_1 = \pm\ \varphi_2/s_2 = \pm\ \varphi_3/s_3$$

break up into pairs of planes. Four of the points S and M coincide with the incenters (1, 1, 1, 1) (1, 1, − 1, − 1) (1, − 1, 1, − 1) (1, − 1, − 1, 1) and the other four form a tetrahedron in fourfold perspective to this. Further details of this case are omitted. It should be noted however that the equifacial tetrahedron, with 3 absolute invariants, gives rise to a configuration which has none; indicating the possibility that the configuration in the general case may not be that of a general set of "associated" points. Another case that can be completely solved is the "isosceles" tetrahedron

$$c_{12} = c_{13} = c_{23} \equiv c, \qquad c_{14} = c_{24} = c_{34} \equiv c'.$$

The net in this case contains three pairs of planes which all belong to the same pencil of cones, which thus have four common generators.

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